# Multi-frequency analysis of electrical ground conductivity data

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**Abstract**

This document presents the general problem of electromagnetic inversion in the frequency domain. The 1-dimensional analysis of a stratified medium is performed at different frequencies; the data measured on the surface (the admittance or the *apparent* electric conductivity) are compared with the solution of the system obtained from a Lagrangian formulation which minimizes a suitable cost function. The information acquired at different frequencies is integrated in the inversion procedure; two approaches are presented and compared: the first method independently solves the problem for each frequency and then integrates the contributions, while the second method solves a global problem which involves all the contributions at the same time. The conjugate gradient algorithm has been used as iterative minimization strategy for both procedures. Numerical examples validate this analysis.

**Keywords:** Electromagnetic analysis, frequency domain, 1-D model, stratified medium, diffusion equation, apparent conductivity, conjugate gradient.

# Contents

# 1 Introduction

The purpose of the analysis here presented is to determine the electric conductivity distribution of a stratified medium solving an inverse problem.

Electromagnetic methods in geophysics utilize fields induced either artificially or by natural sources located, for instance, in the Earth's magnetosphere and ionosphere, typical situation in magnetotellurics.

Starting from the model presented by Dmitriev [1] for magnetotelluric fields, we are able to infer the soil conductivity distribution knowing the frequency dependence of the *apparent conductivity* obtained by illuminating the subsurface with an impulsive electromagnetic source.

This document presents a 1-dimensional analysis. The Lagrangian formulation used to minimize the misfit between measured and simulated data, constraining the propagation to satisfy Dmitriev's model, implies the definition of an *adjoint problem* for the Lagrange multiplier and, consequently, a descent direction which can be used for the iterative procedure leading to the optimal conductivity profile. Data representing the apparent electric conductivity are measured on the surface at different frequencies and then used to reconstruct the conductivity of the medium through this *non linear inversion procedure*.

Two iterative schemes are presented and compared. The first method independently solves the problem for each frequency and then averages the contributions. The second method solves the problem at the same time for all the frequencies.

Section 2 presents some preliminary considerations and the general equations of the propagation of the electromagnetic fields in the frequency domain. The direct and the inverse model are presented in section 3 while the mathematical formulation of the problem is described in section 4. Details about the discretization of the problem and the procedure implemented for solving it are described in section 5. Section 6 shows the non linear inversion problem and a description of the implementaion of the two multi-frequency approaches based on a conjugate gradient algorithm. A critical analysis of the two algorithms is presented in section 7 where positive and negative aspects are discussed. Numerical examples illustrate the performance of the approach here presented.

# 2 Maxwell's equations

Electromagnetic phenomena are governed by the empirical Maxwell's equations. These are uncoupled first-order linear differential equations which can be coupled by the empirical constitutive relations to reduce the number of basic vector field functions. Obviously care must be taken in chosing constitutive relations pertinent to the Earth [2].

In the following the upper case letters will be used for functions expressed in the frequency domain while lower case letters will be used for functions expressed in the time domain.

In the time domain Maxwell's equations are:

$$\nabla \times \mathbf{e} + \frac{\partial \mathbf{b}}{\partial t} = 0, \tag{1}$$

$$\nabla \times \mathbf{h} - \frac{\partial \mathbf{d}}{\partial t} = \mathbf{j}, \tag{2}$$

$$\nabla \cdot \mathbf{b} = 0,$$

$$\nabla \cdot \mathbf{d} = \rho,$$

where:

- **e** is the electric field intensity in $V/m$,
- **b** is the magnetic induction in $Wb/m^2$,
- **h** is the magnetic field intensity in $A/m$,
- **d** is the dielectric displacement in $C/m^2$,
- **j** is the electric current density in $A/m^2$ and
- $\rho$ is the electric charge density in $C/m^3$.

These equations are coupled through the frequency domain constitutive relations:

$$\mathbf{D} = \underline{\varepsilon}(\omega, \mathbf{E}, \mathbf{r}, t, T, P, \ldots) \cdot \mathbf{E},$$

$$\mathbf{J} = \underline{\sigma}(\omega, \mathbf{E}, \mathbf{r}, t, T, P, \ldots) \cdot \mathbf{E},$$

$$\mathbf{B} = \underline{\mu}(\omega, \mathbf{H}, \mathbf{r}, t, T, P, \ldots) \cdot \mathbf{H}.$$

Here the tensors $\underline{\varepsilon}$, $\underline{\mu}$ and $\underline{\sigma}$ describe respectively the dielectric permittivity, the magnetic permeability and the electric conductivity, as a function of angular frequency $\omega$, electric field strength $\mathbf{E}$ or magnetic induction $\mathbf{B}$, position $\mathbf{r}$, time $t$, temperature $T$, pressure $P$ and other parameters.

To build a simplified model we assume isotropy, homogeneity, linearity and temperature-time-pressure independence of the electrical parameters of local regions of the Earth and the magnetic permeability is assumed to be that of the free space: $\mu = \mu_0$. With these assumptions the constitutive lows are:

$$\mathbf{D} = \varepsilon\, \mathbf{E}, \tag{3}$$

$$\mathbf{J} = \sigma\, \mathbf{E}, \tag{4}$$

$$\mathbf{B} = \mu\, \mathbf{H}, \tag{5}$$

where the dielectric permittivity $\varepsilon$ and the electric conductivity $\sigma$ are scalar constant with respect to frequency.

After a 1-dimensional Fourier transformation of equations (1) and (2) and the use of constitutive lows of equations (3), (4) and (5), we can write Maxwell's equations in the frequency domain form:

$$\nabla \times \mathbf{E} + i\mu\omega\mathbf{H} = 0, \tag{6}$$

$$\nabla \times \mathbf{H} - (\sigma + i\varepsilon\omega)\mathbf{E} = 0. \tag{7}$$

Equation (6) and (7) are obviously coupled. Upon taking the curl of each equation and after some substitutions we can derive the following equations:

$$\nabla^2\mathbf{E} + (\mu\varepsilon\omega^2 - i\mu\sigma\omega)\mathbf{E} = 0 \tag{8}$$

and

$$\nabla^2\mathbf{H} + (\mu\varepsilon\omega^2 - i\mu\sigma\omega)\mathbf{H} = 0 \tag{9}$$

(see details of derivation of these formula in [2]).

In the frequency domain these wave equations are known as the Helmholtz equations in **E** and **H**. The term with $\mu\varepsilon\omega^2$, as coefficient, represents the displacement currents term while the term with $\mu\sigma\omega$ is the conduction currents term. For Earth material at frequencies less than $10^5 Hz$ we have that $\varepsilon\omega << \sigma$ and so the displacement currents are much smaller than the conduction currents [2]. In this case equations (8) and (9) become:

$$\nabla^2\mathbf{E} - i\mu\sigma\omega\mathbf{E} = 0$$

and

$$\nabla^2\mathbf{H} - i\mu\sigma\omega\mathbf{H} = 0,$$

while in the case of a stratified medium we found:

$$\frac{\partial^2\mathbf{E}}{\partial z^2} - i\mu\sigma\omega\mathbf{E} = 0, \tag{10}$$

$$\frac{\partial^2\mathbf{H}}{\partial z^2} - i\mu\sigma\omega\mathbf{H} = 0.$$

In the following only equation (10) will be used in order to determine the electric conductivity of the medium.

# 3 Direct and inverse model

According with Dmitriev model derived in the previous section (equation (10)) we consider the following 1-dimensional problem:

$$u''(z) - i\,\omega\,\mu_0\,\sigma(z)\,u(z) = 0, \quad z \in [0, H], \tag{11}$$

$$u(z=0) = 1, \tag{12}$$

$$u'(z=H) - i\,\sqrt{-\,i\,\omega\,\mu_0\,\sigma_H}\;u(z=H) = 0, \tag{13}$$

where condition (12) represents the wave impulse of the electric field while relation (13) is the absorbing condition. Here $u(z) = E_x(z)/E_x(z=0)$, $\omega$ is the angular frequency of the field's variations over time, $\sigma_H = \sigma(z=H)$ and $\mu_0$ is the magnetic permeability.

Surface admittance, as a function of the frequency, measured at $z = 0$ is, by definition, equal to:

$$Y(\omega) = \frac{H_y\,(z=0)}{E_x\,(z=0)} = -\frac{i}{\omega\,\mu_0\,E_x(z=0)} \cdot \left.\frac{\partial E_x}{\partial z}\right|_{z=0} = -\frac{i}{\omega\,\mu_0}\left.\frac{\partial u}{\partial z}\right|_{z=0}. \tag{14}$$

The experimental information commonly used in electromagnetic acquisitions is not the admittance $Y(\omega)$ but the apparent resistivity or the apparent conductivity.

To define these quantities let us consider the following case. If the conductivity of the soil is constant ($\sigma(z) = \sigma$) from the solution of equations (11), (12) and (13) it follows that $Y(\omega) = \sqrt{i\,\sigma\,/\,\omega\,\mu_0}$ and hence:

$$\sigma = \omega\,\mu_0\,|Y(\omega)|^2 = \frac{2\pi\mu_0}{T}\,|Y(\omega = 2\pi/T)|^2, \tag{15}$$

where $T = 2\pi/\omega$ is the oscillation period.

The apparent conductivity $\sigma_{app}(T)$ is the conductivity calculated using equation (15) for every period $T$. It corresponds to the conductivity of a uniform soil whose admittance is $Y(\omega = 2\pi/T)$ for the period $T$. The apparent resistivity $\rho_{app}(T)$ is the inverse of the apparent conductivity.

Given the distribution of the true conductivity $\sigma(z)$ for each frequency $\omega$, the *direct problem* (defined by equations (11), (12) and (13)) determines the component of the electric field $E_x(z)$ along the $x$-axis and so the value of function $u(z),\ z \in [0, H]$. The admittance $Y(\omega)$ can be finally calculated using equation (14).

On the other hand, the purpuse of the *inverse problem* is the determination of the electrical conductivity distribution $\sigma(z)$ from the known frequency dependence of the admittance $Y(\omega)$ on the electromagnetic field measured at the Earth's surface ($z = 0$).

Unfortunatelly the resulting inverse problem is not well posed: existence and uniqueness of the solution is not guaranteed and the solution, if existing, may not be continuously dependent on data.

It can be shown that the recovering of the integrated conductivity $S(\xi) = \int_0^\xi \sigma(z)dz$ is a well posed problem, but then, as this equation is an integral Legendre equation of the first type, the recovering of $\sigma$ from $S$ is ill posed (see [1]).

## 4 Mathematical analysis

The minimization of the misfit between measured and simulated data has been formulated as a constrained minimization problem with Lagrangian multiplier. The cost function considered is

$$\mathcal{C}[u] = \frac{1}{2} \int_0^H \left( \ln \frac{|u'(z)|^2}{\omega\, \mu_0\, \sigma_{app}} \right)^2 \delta(z)\, dz,$$

which compares the logarithm of two apparent conductivities; the function $\delta(z)$ is defined in this way, for all $a$ and $b$, with $a \neq b$:

$$\int_a^b \delta(z)\, dz = \begin{cases} 1 & \text{if } 0 \in [a, b] \\ 0 & \text{otherwise} \end{cases} . \tag{16}$$

The Lagrangian functional is written as follow:

$$\mathcal{L}[u, \lambda, \sigma] = \mathcal{C}[u] + Re\left\{ \int_0^H \overline{\lambda(z)}\, (u''(z) - i\, \omega\, \mu_0\, \sigma(z)\, u(z))\, dz \right\} = \tag{17}$$

$$= \mathcal{C}[u] + Re\left\{ \int_0^H u(z)\, (\overline{\lambda''(z)} - i\, \omega\, \mu_0\, \sigma(z)\, \overline{\lambda(z)})\, dz \right\}, \tag{18}$$

where $\lambda(z)$ is a complex Lagrangian multiplier for which two additional conditions are required:

$$\overline{\lambda'(0)}\, u(0) \;=\; \overline{\lambda(0)}\, u'(0), \tag{19}$$

$$\overline{\lambda'(H)}\, u(H) \;=\; \overline{\lambda(H)}\, u'(H). \tag{20}$$

Those boundary conditions are derived imposing that the surface terms coming from the integration by part of the integrand, are zero. Thus if we differentiate the functional $\mathcal{L}$ with respect to $\lambda(z)$, $u(z)$ and $\sigma(z)$, requiring the minimum condition, we obtain the following non linear system:

$$
\begin{cases}
\dfrac{\partial \mathcal{L}}{\partial \overline{\lambda(z)}} = u''(z) - i\,\omega\,\mu_0\,\sigma(z)\,u(z) = 0 \\[2ex]
\dfrac{\partial \mathcal{L}}{\partial u(z)} = \left( \dfrac{2\,Re\,\{u'(0)\}}{|u'(0)|^2}\; ln \dfrac{|u'(0)|^2}{\omega\,\mu_0\,\sigma_{app}} \right) + \left( \overline{\lambda''(z)} - i\,\omega\,\mu_0\,\sigma(z)\,\overline{\lambda(z)} \right) = 0 \\[2ex]
\dfrac{\partial \mathcal{L}}{\partial \sigma(z)} = \dfrac{d\,\mathcal{C}}{d\sigma(z)} = -\,Re\,\{i\,\omega\,\mu_0\,u(z)\,\overline{\lambda(z)}\}.
\end{cases}
\tag{21}
$$

From the first equation of system (21), it is possible, for a given $\sigma$, to determine the function $u(z)$ which provides the right hand side of the equation for $\lambda$. The equation for $\lambda$ with conditions (19) and (20) is called the *adjoint problem* and the solution $\lambda$ is the *adjoint field*. The third equation is used as definition of the descent direction $-\frac{d\mathcal{C}}{d\sigma(z)}$ in the iterative algorithm implemented for the minimization of the cost function $\mathcal{C}$ with respect to $\sigma$. In our case the algorithm implemented is the non linear conjugate gradient.

# 5 Discretization

The soil is considered divided into $M$ layers with the same thickness, as illustrated in figure (1); the electric conductivity is assumed to be a piecewise constant function with values $\sigma_k$ where $z_{k-1} \leq z < z_k$ and $k \in \{1, 2, ..., M\}$.

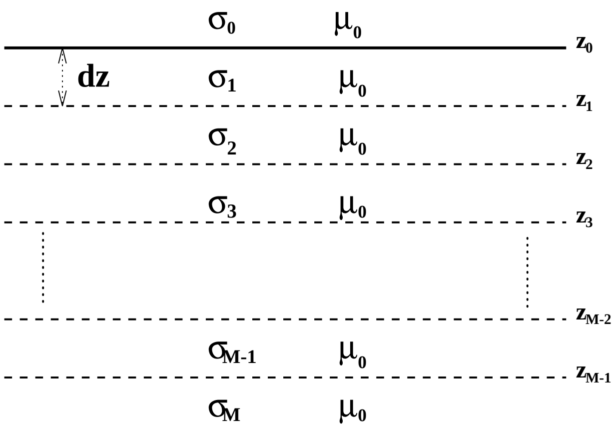


Figure 1: *Stratification of the soil layers.*

From a mathematical point of view, the first equation of system (21) is discretized using a finite difference scheme; the second equation is discretized with a finite element scheme. The inverse problem is solved by an iterative procedure presented and discussed in this section.

The equation for $u(z)$ can be discretized imposing on the boundary $z = 0$ a condition on the solution and on the boundary $z = H$ a condition on the first derivative (condition (12) and (13)); the finite difference approximation gives rise for each value of $\omega_j$, $j = 1, 2, ..., N_f$, to the following linear system:

$$
\begin{cases}
u_0 = 1 \\
\dfrac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + i\,\omega_j\,\mu_0\,\sigma_k\,u_k = 0, \quad k = 1, \ldots, M-1 \\
\dfrac{u_M - u_{M-1}}{h} = i\,\sqrt{i\,\omega_j\,\mu_0\,\sigma_M}\,u_M.
\end{cases}
\tag{22}
$$

In the resulting system,

$$
\mathbf{Au} = \mathbf{b}, \tag{23}
$$

$\mathbf{A} = \mathbf{A}(\sigma, \omega)$ is the matrix of the coefficients, $\mathbf{u}$ is the vector of the unknowns and $\mathbf{b}$ is the right hand side. This system is complex and tridiagonal and it is solved by simply decomposition of the matrix $\mathbf{A}$ into triangular matrices and by solving consequently two triangular systems.

The system for the function $\lambda(z)$ is described by these three equations:

$$
\begin{cases}
\overline{\lambda'(0)}\,u(0) \;=\; \overline{\lambda(0)}\,u'(0) \\
\overline{\lambda''(z)} - \;i\,\omega_j\,\mu_0\,\sigma(z)\,\overline{\lambda(z)} \;= rhs(z)\,\delta(z) \\
\overline{\lambda'(H)}\,u(H) \;=\; \overline{\lambda(H)}\,u'(H),
\end{cases}
\tag{24}
$$

where the term

$$
rhs(z) \;=\; \frac{2\,Re\,\{u'(z)\}}{|u'(z)|^2}\;ln\left(\frac{|u'(z)|^2}{\omega_j\,\mu_0\,\sigma_{app}}\right)
$$

gives a contribution only at $z = 0$. Using the information about the vector solution $\mathbf{u}(z)$, previously determined by solving system (23), we can modify the first and the last equation of system (24) in this way:

$$
\overline{\lambda'(0)} = u'(0)\,\overline{\lambda(0)},
$$

$$
\overline{\lambda'(H)} = i\,\sqrt{-\,i\,\omega_j\,\mu_0\,\sigma(H)}\,\overline{\lambda(H)}.
$$

To properly handle the right hand side of (24), a linear finite element approximation has been implemented according to the following standard finite elements analysis.

After multiplying by a suitable function $\phi(z)$, we can integrate the second equation of system (24) with respect to the variable $z$ between zero and the value $H$:

$$
\int_0^H \overline{\lambda''(z)}\,\phi(z)\,dz - \;i\,\omega_j\,\mu_0 \int_0^H \sigma(z)\,\overline{\lambda(z)}\,\phi(z)\,dz = \int_0^H rhs(z)\,\delta(z)\,\phi(z)\,dz.
$$

If we select the test function $\phi(z)$ such that $\phi(0) = \phi(H) = 0$, and we integrate by part, we obtain:

$$
\overline{\lambda'(H)} - \overline{\lambda'(0)} - \int_0^H \overline{\lambda'(z)}\,\phi'(z)\,dz - \;i\,\omega_j\,\mu_0 \int_0^H \sigma(z)\,\overline{\lambda(z)}\,\phi(z)\,dz = \int_0^H rhs(z)\,\delta(z)\,\phi(z)\,dz.
$$

We discretize the problem in the variable $z$ so that the function $\lambda(z)$ is defined in this way:

$$
\overline{\lambda(z)} = \sum_{k=0}^{M} \overline{\lambda_k}\,\phi_k(z) \qquad \text{and} \qquad \overline{\lambda'(z)} = \sum_{k=0}^{M} \overline{\lambda_k}\,\phi_k'(z).
$$

For a generic layer $k = 1, \ldots, M$ we have the following formulation:

$$\overline{\lambda'(H)} - \overline{\lambda'(0)} - \left[ \int_{z_{k-1}}^{z_k} \overline{\lambda_{k-1}} \phi'_{k-1} \phi'_k dz_k + 2 \int_{z_k}^{z_{k+1}} \overline{\lambda_k} \phi'_k \phi'_k dz_k + \int_{z_{k+1}}^{z_{k+2}} \overline{\lambda_{k+1}} \phi'_{k+1} \phi'_k dz_k \right] +$$

$$-i\omega_j \mu_0 \int_{z_k}^{z_{k+1}} \sigma_k \overline{\lambda_k} \phi_k \phi_k dz_k = \sum_{k=0}^{M-1} \int_{z_k}^{z_{k+1}} rhs(z) \delta(z) \phi_k dz_k,$$

where $dz_k$ is the tickness of the $k$-th layer. In our model $dz_k$ is constant and equal to $h$ for all $k$. If we chose the linear test function $\phi(z)$ such that:

$$\phi_k(z) = \begin{cases} \dfrac{z - z_i}{z_{i+1} - z_i} & \text{if } z \in [z_i, z_{i+1}] \\[2ex] \dfrac{z_i - z}{z_i - z_{i-1}} & \text{if } z \in [z_{i-1}, z_i] \end{cases} ,$$

the discretization becomes:

$$\begin{cases} \dfrac{\overline{\lambda_0} - \overline{\lambda_1}}{h} - \, i \, \omega_j \, \mu_0 \, \sigma_0 \, \overline{\lambda_0} \, h \; = \; \dfrac{rhs(0)}{2} \; + \; \overline{\lambda_0} \, u'_0 \\ \dfrac{\overline{\lambda_{k-1}} - 2\overline{\lambda_k} + \overline{\lambda_{k+1}}}{h} - \, i \, \omega_j \, \mu_0 \, \sigma_k \, \overline{\lambda_k} \, h \; = \; 0 \qquad k = 1, ..., M-1 \\ \dfrac{\overline{\lambda_{M-1}} - \overline{\lambda_M}}{h} - \, i \, \omega_j \, \mu_0 \, \sigma_M \, \overline{\lambda_M} \, h \; = \; -i \, \sqrt{- \, i \, \omega_j \, \mu_0 \, \sigma_M} \; \overline{\lambda_M}. \end{cases} \tag{25}$$

The system (25) is a linear system

$$\mathbf{B}\vec{\lambda} = \mathbf{d} \tag{26}$$

where $\mathbf{B} = \mathbf{B}(\sigma, \omega)$, $\vec{\lambda}$ is the vector of the unknowns and $\mathbf{d}$ is the right hand side. Even in this case the system is complex and tridiagonal; it is solved by simply decomposing the matrix $\mathbf{B}$ into triangular matrices and by solving consequently two trangular systems.

If we consider the third relation of system (21), which describes the descent direction, we can now simply evaluate it, once the vector $\mathbf{u}$ and the vector $\overline{\lambda}$ are known, for each $k = 1, ..., M$:

$$\mathbf{g}_j = \omega_j \, \mu_0 \, Re\{i \, \mathbf{u}_j \, \mathbf{I_d} \, \overline{\lambda}_j\}, \tag{27}$$

where $\mathbf{I_d}$ stands for the identity matrix.

# 6 Inversion technique

In order to determine the optimal conductivity vector $\sigma$, the iterative procedure of the conjugate gradient has been implemented. Starting from an initial feasible guess $\sigma^{(0)}$ the algorithm evaluates the descent direction solving first the system (23) for the vector $\mathbf{u}$ and then that for the vector $\vec{\lambda}$, system (26). The descent direction $\mathbf{g}$, described in equation (27), is used by the iterative procedure minimizing the cost function $\mathcal{C}$ which is obviously characterized by a large number of local minima.

For each frequency $\omega_j$ the procedure can be summarized as follow:

- given the admittance $Y(\omega_j)$ or the apparent conductivity $\sigma_{app}$ and an initial guess $\sigma^{(0)}$, the direction $\mathbf{g}^{(0)}$ is calculated using definition (27);

- the conjugate initial direction $\mathbf{d}^{(0)}$ is set equal and opposite to $\mathbf{g}^{(0)}$:

$$\mathbf{d}^{(0)} = -\mathbf{g}^{(0)},$$

  while, for a generic iteration $n+1$, the direction is calculated as a linear combination of the gradient and the direction at the $n$-th iteration:

$$\mathbf{d}^{(n+1)} = -\mathbf{g}^{(n+1)} + \beta\,\mathbf{d}^{(n)};$$

- the line search parameter has been implemented in order to determine the solution which minimizes the cost function $\mathcal{C}$; the parameter $\alpha$ is the optimal parameter; the generic solution at step $n+1$ is given by:

$$\sigma^{(n+1)} = \sigma^{(n)} + \alpha\,\mathbf{d}^{(n+1)}.$$

The choice of the parameter $\beta$ can be done using the definition of Fletcher and Reeves:

$$\beta_{FR}^{(n)} = \frac{||\mathbf{g}^{(n)}||^2}{||\mathbf{g}^{(n-1)}||^2},$$

or using the definition of Polak and Rebière:

$$\beta_{PR}^{(n)} = \frac{\mathbf{g}^{(n)T}\left(\mathbf{g}^{(n)} - \mathbf{g}^{(n-1)}\right)}{||\mathbf{g}^{(n-1)}||^2}.$$

Both the choices of the parameter $\beta$ are equivalent for our algorithm without any evident differences.

To perform the line search for the parameter $\alpha$, a golden section algorith has been implemented. The parameter is the one which minimizes the cost function.

Stop criteria for the conjugate gradient iterations are:

- $\left\| \dfrac{\sigma^{(n+1)} - \sigma^{(n)}}{\sigma^{(0)}} \right\| < \varepsilon$, which controls the error in the solution;
- $\left\| \dfrac{\mathbf{g}^{(n+1)}}{\mathbf{g}^{(0)}} \right\| < \varepsilon$, which controls the error in the gradient;
- $\left| \dfrac{\mathcal{C}^{(n+1)} - \mathcal{C}^{(n)}}{\mathcal{C}^{(0)}} \right| < \varepsilon$, which controls the cost function.

If these conditions are not reached a control of the number of iterations has been also implemented; if the number of iterations reaches the maximum admissible value, the algorithm will stop.

# 7 Validation of the algorithm

The general scheme of the algorithm described in the previous section has been implemented for the multifrequency analysis of the problem; synthetic data representing the apparent conductivity were simulated for different values of the frequency.

In order to process the information relative to the different frequencies, two versions of the algorithm have been examined:

- the *average gradient* (*AG*) method and
- the *average solution* (*AS*) method.

They differ each other in the construction of the intermediate solution $\sigma$ supplied by the conjugate gradient.

Given $N_f$ frequencies, the first version of the algorithm is a plain implementation of system (21) where the cost function is the sum of the cost functions for every frequency:

$$\mathcal{C} = \sum_{j=1}^{N_f} \mathcal{C}(\omega_j).$$

With this approach the global direction is:

$$\frac{d\mathcal{C}}{d\sigma} = -\mu_0 \sum_{j=1}^{N_f} \omega_j \, Re\{i\, \mathbf{u}(\omega_j)\, \mathbf{I_d}\, \overline{\lambda}(\omega_j)\} = \sum_{j=1}^{N_f} \mathbf{g}(\omega_j). \tag{28}$$

Here the dependence from the depth $z$ has been omitted.

Once an initial guess has been chosen, the second version of the algorithm, the *AS* method, solves $N_f$ independent systems (21), one for each frequency $\omega_j$. The intermediate solution $\sigma_{\mathbf{j}}^{(\mathbf{m})}$ of every system $j = 1, \dots, N_f$, determines a new initial guess for the algorithm in the sense that the average of these solutions, calculated at the previous step, is used as the initial guess for the following step:

$$\sigma^{(\mathbf{m+1})} = \frac{1}{N_f} \sum_{j=1}^{N_f} \sigma_{\mathbf{j}}^{(\mathbf{m})},$$

where $\sigma^{(\mathbf{m+1})}$ is used as initial guess for each partial problem: $\sigma_{\mathbf{j}}^{(\mathbf{m+1})} = \sigma^{(\mathbf{m+1})}$ for all $j = 1, \dots, N_f$. The flowchart of the algorithm is shown in figure (2).

From a mathematical point of view the *AS* algorithm controls the sum of the minimized costs for each frequency ($\sum_\omega \min_\sigma \mathcal{C}(\omega, \sigma)$) while the *AG* version controls the minimum of the sum of the costs ($\min_\sigma \sum_\omega \mathcal{C}(\omega, \sigma)$).

Both algorithms are very sensitive to the frequency sampling; very high frequencies will give an accurate description of the conductivity profile near the surface while low frequencies will allow the electromagnetic field to penetrate deeper and then provide a more reliable information of the conductivity for large $z$.

We observe that the *AS* method is more accurate than the *AG* method. In figure (3) a synthetic curve of conductivity is compared with the solution computed with the *AS* algorithm and the *AG* algorithm.

The range of frequency is $[10Hz, 10^4Hz]$ and 256 samples have been used. The soil has been discretized with 16 layers till the depth of 3 m. The initial solution has been set equal to the constant value of 50 mS/m.

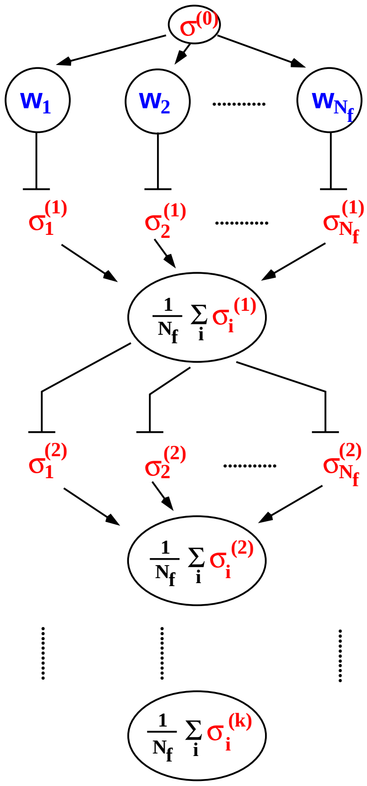


Figure 2: *The* AS *algorithm.*

Among the two curves it is possible to observe that the *AG* method gives a good accuracy for shallow and deep layers while the *AS* method performs correctly in the near surface zone and not very well in the deeper one where the solution does not deviate from the initial constant guess of $50mS/m$.

# 8 Conclusion

In this document the electromagnetic inverse problem for the reconstruction of the conductivity profile of the soil has been formulated and developed.

Starting from the physical description of the problem (using Maxwell's equations), the direct and the inverse model have been defined. The mathematical analysis makes use of the Lagrangian minimization of a suitable cost function measuring the discrepancy between real and simulated data; the discretization of the problem with finite elements and finite differences determines $N_f$ independent linear systems, where $N_f$ is the number of different frequencies used for data acquisition on the surface.

The inversion technique in combined with an iterative scheme implementing a non linear conjugate gradient strategy; two different algorithms have been developed in order to find the solution which combines the contributions coming from the different frequencies of acquisition.

At the present the precedure has not yet tested on real datasets.

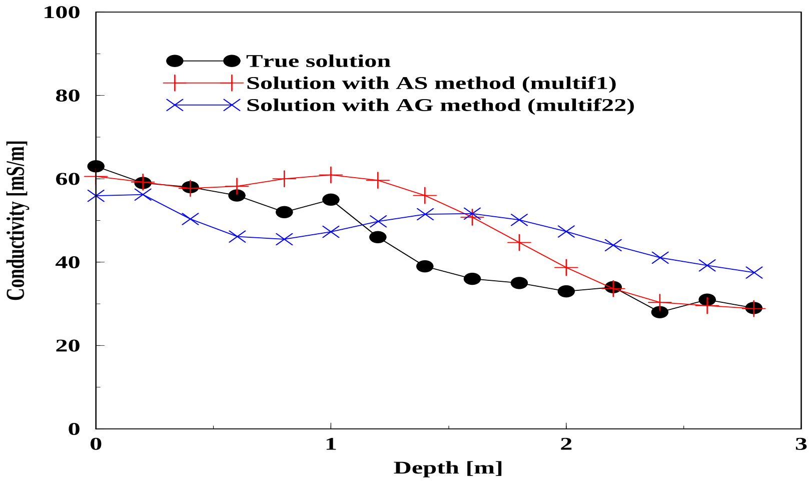


Figure 3: *The synthetic curve of conductivity is compared with the solutions of the* AS *method and the* AG *method; 256 frequencies have been used and the number of unknowns is 16.*

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